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REMARKS ON THE FIRST MAIN THEOREM IN EQUIDISTRIBUTION THEORY. III

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1. On C^n, there are two canonical convex exhaustions, $\tau_0 = \Sigma_i z_i \bar{z}_i$ and $\tau_1 = \log(1 + \Sigma_i z_i \bar{z}_i)$. Up to scalar factors, $dd^c\tau_0$ is the kähler form of the flat metric on C^n, and $dd^c\tau_1$ is the kähler form of the pull-back of the Fubini-Study metric by the inclusion $C^n \subseteq P_nC$. (We also call the latter metric the *Fubini-Study metric on* C^n.) Considering the way $dd^c\tau$ enters into the condition which guarantees that the complement of the image under a holomorphic $f\colon C^n \to M$ be of measure zero (Theorem 5.1, Part II [13]), we readily appreciate the importance of these exhaustion functions. It seems that much attention has thus far been lavished on τ_0, and this is unjustified—and perhaps unjustifiable. For instance, the inclusion $C^n \subseteq P_nC$, which is of course *quasi-surjective* ($\equiv$ surjective up to a set of measure zero), does not satisfy the condition stipilated in equation (12) of Corollary 5.2 in Part II [13]. What we would like to contend here is that τ_1 should be given a more prominent position than τ_0 in equidistribution theory. The critical feature of τ_1 which accounts for its usefulness is that $\int_{C^n} (dd^c\tau_1)^n$ is finite, i.e., the Fubini-Study metric induces a totally finite measure on C^n. The theorems proved in this note, all of them intuitively plausible, purport to demonstrate this fact.

There is nevertheless one place where the flat metric on C^n (and hence τ_0) might prove to be useful. This is the case of a holomorphic $f\colon C^n \to C^n$; the situation is most natural (e.g. the Faton-Bieberbach example) and it seems legitimate that one should capitalize on the simplicity of the flat metric by carrying out all investigations in terms of it. The usual procedure of first imbedding C^n into P_nC and then considering the composite map $f\colon C^n \to P_nC$ becomes in this light less desirable. For this reason, a proof, directly exploiting the flat metric, of the first main theorem with C^n as image manifold is given in the appendix. It turns out that no harmonic theory is needed for this case, but the theorem so obtained is not very strong. Time will decide whether further work should be done in this direction, or whether this undertaking should be abandoned altogether.

2. Let us begin by recalling some elementary facts about C^n and P_nC. On

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C^n the flat metric is

$$ds_0^2 = \Sigma_i \, dz_i \otimes d\bar{z}_i \ ,$$

whose kähler form is denoted by

$$(1) \qquad \omega_0 = \frac{\sqrt{-1}}{2} \Sigma_i \, dz_i \wedge d\bar{z}_i \ .$$

With $\tau_0 = \Sigma_i \, z_i \bar{z}_i$, we have

$$(2) \qquad dd^c \tau_0 = 4\omega_0 \ .$$

The Fubini-Study metric on C^n, i.e., the pull-back of the Fubini-Study metric on $P_n C$ by the inclusion $C^n \subseteq P_n C$, is

$$ds_1^2 = \frac{(1 + \Sigma_i z_i \bar{z}_i)\Sigma_i \, dz_i \otimes d\bar{z}_i - (\Sigma_i \, \bar{z}_i dz_i) \otimes (\Sigma_i \, z_i d\bar{z}_i)}{(1 + \Sigma_i \, z_i \bar{z}_i)^2} \ ,$$

whose kähler form is denoted by

$$(3) \qquad \omega_1 = \frac{\sqrt{-1}}{2} \frac{(1 + \Sigma_i z_i \bar{z}_i)\Sigma_i \, dz_i \wedge d\bar{z}_i - (\Sigma_i \, \bar{z}_i dz_i) \wedge (\Sigma_i \, z_i d\bar{z}_i)}{(1 + \Sigma_i \, z_i \bar{z}_i)^2} \ .$$

With $\tau_1 = \log (1 + \Sigma_i \, z_i \bar{z}_i)$, we see that

$$(4) \qquad dd^c \tau_1 = 4\omega_1 \ .$$

Note that the volume elements of ds_0^2, ds_1^2 are, respectively,

$$(5) \qquad \Psi_0 = \frac{\omega_0^n}{n!} = \left(\frac{\sqrt{-1}}{2}\right)^n (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n) \ ,$$

$$(6) \qquad \Psi_1 = \frac{\omega_1^n}{n!} = \frac{1}{(1 + \Sigma_i z_i \bar{z}_i)^{n+1}} \Psi_0 \ .$$

Consequently,

$$(7) \qquad \int_{C^n} \omega_1^n = \pi^n \ .$$

From (2) and (4), it follows that $dd^c \tau_0$ and $dd^c \tau_1$ are everywhere positive definite, so that Theorem 5.1 of Part II [13] can be applied.

Next suppose a holomorphic mapping $f\colon M_1 \to M_2$ is given, where M_1 and M_2 are n-dimensional kähler manifolds with kähler forms κ_1 and κ_2 respectively. We wish to express the $2n$-forms $f^*\kappa_2^{n-1} \wedge \kappa_1$ and $f^*\kappa_2^n$ in terms of κ_1^n. To this end, let $f(p) = q$, and $e_1, \cdots, e_n$ and $\mu^1, \cdots, \mu^n$ be local orthonormal frame and dual coframe of type (1, 0) around p. Correspondingly, let $f_1, \cdots, f_n$ and $\nu^1, \cdots, \nu^n$ be local orthonormal frame and dual coframe of type (1, 0) around q. If $df(e_i) = \Sigma_j a_i^j f_j$, then $f^*(\nu^i) = \Sigma_j a_j^i \mu^j$. Around p and q,

$$\kappa_1 = \frac{\sqrt{-1}}{2} \Sigma_i \mu^i \wedge \bar{\mu}^i \,, \qquad \kappa_2 = \Sigma_i \frac{\sqrt{-1}}{2} \nu^i \wedge \bar{\nu}^i \,,$$

so that

$$f^*(\kappa_2) = \frac{\sqrt{-1}}{2} \Sigma_{k,l} (\Sigma_i \, a_k^i \bar{a}_l^i) \mu^k \wedge \bar{\mu}^l \,.$$

Put $C_{kl} = \Sigma_i \, a_k^i \bar{a}_l^i$. Then C is a positive semi-definite hermitian matrix, and has the following geometric interpretation: If $\langle , \rangle_1$ and $\langle , \rangle_2$ are the kähler metrics of M_1 and M_2, then $\langle e_k, e_l \rangle_1 = \delta_{kl}$ and $\langle df(e_k), df(e_l) \rangle_2 = C_{kl}$. Being positive, semi-definite and hermitian, C can be diagonalized by a unitary matrix; let its eigenvalues be arranged in an increasing sequence:

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_n \,,$$

and we may assume $\{e_i\}$, $\{\mu^i\}$, and $\{f_i\}$, $\{\nu^i\}$ have been so chosen that

$$\begin{aligned} &\langle df(e_i), df(e_i) \rangle_2 = \lambda_i \,, \\ &f^*\kappa_2 = \frac{\sqrt{-1}}{2} \Sigma_i \, \lambda_i \mu^i \wedge \bar{\mu}^i \,. \end{aligned} \tag{8}$$

Hence

$$\begin{aligned} f^*\kappa_2^{n-1} \wedge \kappa_1 \\ &= \left(\frac{\sqrt{-1}}{2}\right)^n (n-1)! (\Sigma_i \, \lambda_1 \cdots \hat{\lambda}_i \cdots \lambda_n)(\mu^1 \wedge \bar{\mu}^1) \wedge \cdots \wedge (\mu^n \wedge \bar{\mu}^n) \,, \\ f^*\kappa_2^n &= \left(\frac{\sqrt{-1}}{2}\right)^n n! (\lambda_1 \cdots \lambda_n)(\mu^1 \wedge \bar{\mu}^1) \wedge \cdots \wedge (\mu^n \wedge \bar{\mu}^n) \,, \end{aligned}$$

where $\hat{\lambda}_i$ indicates that λ_i is omitted in the product. Let us introduce the notation:

$$\begin{aligned} \sigma_{n-1} &= \Sigma_i \, \lambda_i \cdots \hat{\lambda}_i \cdots \lambda_n \,, \\ \sigma_n &= \lambda_1 \cdots \lambda_n \,, \end{aligned}$$

i.e., σ_{n-1} and σ_n are the $(n-1)$-th and n-th elementary symmetric functions of $\lambda_1, \cdots, \lambda_n$ respectively. Then

$$\begin{aligned} f^*\kappa_2^{n-1} \wedge \kappa_1 &= \frac{1}{n} \sigma_{n-1} \kappa_1^n \,, \\ f^*\kappa_2^n &= \sigma_n \kappa_1^n \,. \end{aligned} \tag{9}$$

(In general, if σ_i denotes the i-th elementary symmetric function of $\lambda_1, \cdots, \lambda_n$, then $f^*\kappa_2^i \wedge \kappa_1^{n-i} = \binom{n}{i}\sigma_i\kappa_1^n$.) We also recall the well-known inequality:

$$\left(\frac{1}{n}\sigma_{n-1}\right)^{1/(n-1)} \geq \sigma_n^{1/n} . \tag{10}$$

Each λ_i is a globally defined continuous function on M_1 and is obviously an invariant of f. Let us agree to call it the *i-th eigenfunction of f*. Similarly, we call σ_{n-1} and σ_n respectively the $(n-1)$-th and n-th *elementary symmetric functions of f*. Also note that $df\colon (M_1)_p \to (M_2)_q$ is a homomorphism between inner product spaces and so has a norm $\| df \|_p$. Then (8) implies that $\| df \|_p = \sqrt{\lambda_n(p)}$. In other words, *$\sqrt{\lambda_n}$ is the norm of the differential of f.*

3. The first theorem is an immediate consequence of Theorem 5.1 of Part II [13] and (4) of § 2.

Theorem 1. *Let $f\colon C^n \to M$ be a holomorphic mapping, where M is a compact kähler manifold of dimension n, and df is nonsingular somewhere. Let $C_r^n = \{z\colon \Sigma_i z_i\bar{z}_i \leq e^r - 1\}$, and ω_1(resp. κ) be the kähler form of the Fubini-Study metric on C^n (resp. of the kähler metric on M). If*

$$\liminf_{r\to\infty} \frac{\int_{C_r^n} f^*\kappa^{n-1} \wedge \omega_1}{\int_0^r dt \int_{C_t^n} f^*\kappa^n} = 0 , \tag{11}$$

then f is quasi-surjective i.e., $M - f(C^n)$ is of measure zero.

The quantity $\lim_{r\to\infty} \int_0^r dt \int_{C_t^n} f^*\kappa^n$ is obviously infinite if df is nonsingular somewhere. So if we assume that $\| df \|_p$ is bounded by a constant A independent of p with respect to the Fubini-Study metric or C^n and the given kähler metric on M, (in which case we say f is of *bounded distortion* with respect to the said metrics), then σ_{n-1} is also bounded by nA^{n-1} on C^n. Consequently, by virtue of (7) and (9), $\int_{C^n} f^*\kappa^{n-1} \wedge \omega_1$ is bounded, and so (11) is automatically satisfied. Hence,

Theorem 2. *Let C^n be equipped with the Fubini-Study metric, and $f\colon C^n \to M$ be a holomorphic mapping of bounded distortion into a compact kähler manifold of dimension n such that df is nonsingular somewhere. Then f is quasi-surjective.*

Of course, Theorem 2 holds under the weaker assumption that σ_{n-1} is a bounded function on C^n. Now the condition of bounded distortion has a very simple geometric meaning: $\|df\| \leq A$ if and only if the image of the unit sphere in each $C_p^n (p \in C^n)$ is contained in the unit sphere of radius A in $M_{f(p)}$. We can interpret this same condition differently in a special case: the differential of the inclusion $i: C^n \subseteq P_nC$ has norm one with respect to the Fubini-Study metrics, so that a mapping of bounded distortion $f: C^n \to P_nC$ may be regarded as a "uniformly bounded deformation" of i. One's intuition says that such a mapping should remain quasi-surjective, and Theorem 2 tells us that this is indeed the case. In Corollary 4 below, we will encounter a second kind of "uniformly bounded deformation" of i, and again quasi-surjectivity prevails.

Theorem 3. *Let C^n be given the Fubini-Study metric, and $f: C^n \to M$ a holomorphic mapping into a compact kähler manifold of dimension n so that df is nonsingular somewhere. Assume there is a constant K such that the elementary symmetric functions of f satisfy*

$$\left(\frac{1}{n}\sigma_{n-1}\right)^{n/(n-1)} \leq K\sigma_n . \tag{12}$$

Then f is quasi-surjective.

Remark. (12) is, up to constant factor, the reverse of the universal inequality (10).

Proof. As usual, we will try to show (11) is satisfied. By (9) and Hölder's inequality,

$$\int_{C_t^n} f^*\kappa^{n-1} \wedge \omega_1 = \int_{C_t^n} \frac{1}{n}\sigma_{n-1}\omega_1^n$$
$$\geq \left[\int_{C_t^n} \left(\frac{1}{n}\sigma_{n-1}\right)^{n/(n-1)} \omega_1^n\right]^{(n-1)/n} \left(\int_{C_t^n} \omega_1^n\right)^{1/n} .$$

By (7), (9) and (12), we have

$$\int_{C_t^n} f^*\kappa^{n-1} \wedge \omega_1 \leq \pi K\left(\int_{C_t^n} \sigma_n\omega_1^n\right)^{(n-1)/n} = \pi K\left(\int_{C_t^n} f^*\kappa^n\right)^{(n-1)/n} .$$

Let $g(t) = \int_{C_t^n} f^*\kappa^{n-1} \wedge \omega_1$; then (11) is implied by

$$\liminf_{r\to\infty} \frac{g(r)}{\int_0^r g^{n/(n-1)}} = 0 .$$

Let $F(r) = \int_0^r g^{n/(n-1)}$; then it suffices to prove

$$\liminf_{r\to\infty} \frac{(F')^{(n-1)/n}(r)}{F(r)} = 0 ,$$

or equivalently,

$$\liminf_{r\to\infty} \frac{F'(r)}{F^{n/(n-1)}(r)} = 0 .$$

Now since df is nonsingular somewhere, F is a strictly positive and strictly increasing once differentiable function on $[1, \infty)$. Hence a well-known lemma in the classical Nevanlinna theory (e.g. Lemma 7.2 of [8]) implies that if k is a real number such that $1 < k < \frac{n}{n-1}$, then $F'(r) \leq F^k(r)$ except on an open set I of $[1, \infty)$ with the property that $\int_I d \log x < \infty$. So,

$$0 \leq \liminf_{r\to\infty} \frac{F'(r)}{F^{n/(n-1)}(r)} \leq \liminf_{r\to\infty} \frac{1}{[F(r)]^{n/(n-1)-k}} = 0 ,$$

because $\frac{n}{n-1} - k > 0$ and $F(r) \to +\infty$ as $r \to \infty$. q.e.d.

We can now give a geometric interpretation of the above result. First, consider a definition: A holomorphic mapping $f: M_1 \to M_2$ between kähler manifolds M_1, M_2 is said to be *quasi-conformal* (*of order* K) if and only if at each $p \in M_1$ the ratio of the longest axis to the shortest axis of the hyper-ellipsoid $df(S_p)$ in the tangent space $(M_2)_{f(p)}$ (where S_p is the unit sphere in the tangent space $(M_1)_p$) is bounded by K. So let C^n be equipped with the Fubini-Study metric, and $f: C^n \to M$ quasi-conformal of order K so that $\sqrt{\lambda_n/\lambda_1} \leq K$ by using the notation of § 2. Hence,

$$\left(\frac{1}{n}\sigma_{n-1}\right)^{1/(n-1)} \leq \lambda_n \leq K^2\lambda_1 \leq K^2(\sigma_n)^{1/n} ,$$

and so (12) is satisfied.

Corollary 4. *Let C^n be given the Fubini-Study metric, and $f: C^n \to M$ a quasi-conformal mapping into a compact kähler manifold of dimension n. If df is nonsingular somewhere, then f is quasi-surjective.*

It is a bit surprising that it takes such sophisticated methods to prove such naive statements as Theorem 2 and Corollary 4. Another application of quasi-conformal holomorphic mappings may be found in [14].

Appendix

We will consider exclusively holomorphic mappings into C^n, so let us first introduce certain forms on C^n. In the same notation as in § 2, let $z = (z_1, \cdots, z_n) \in C^n$, and let $\|z\| = (\Sigma_i z_i\bar{z}_i)^{1/2}$ denote the usual euclidean norm of C^n. Define (see (5))

$$(13)\quad \begin{aligned} \theta &= \frac{(n-2)!}{4\pi^n}\left[\frac{1}{(1+\|z\|^2)^{n-1}} - 1\right]\Psi_0 \,, \\ \Theta &= \frac{n!}{\pi^n}\frac{1}{(1+\|z\|^2)^{n+1}}\Psi_0 \,. \end{aligned}$$

Then a simple computation gives

$$\Delta\theta = \Theta \,,$$

where Δ denotes the general Laplace operator with respect to the flat metric of C^n. On functions, it should be pointed out that $\Delta = -\frac{1}{4}\Sigma_i \frac{\partial^2}{\partial z_i \partial \bar{z}_i}$ in conformity with the sign convention in de Rham's book [5]. Also, (7) implies

$$(14)\quad \int_{C^n} \Theta = 1 \,.$$

Let $a = (a_1, \cdots, a_n) \in C^n$, and δ_a be the Dirac measure at a. If $\xi_a = \frac{-1}{S_{2n}(2n-2)} \frac{1}{\|z-a\|^{2n-2}} \Psi_0$, where $S_{2n} = \frac{2\pi^n}{(n-1)!}$ is the volume of the unit sphere in C^n, then the classical fact about the fundamental solution of the Laplacian translates into

$$(15)\quad \Delta\xi_a = -\delta_a \,.$$

Thus if $\Pi_a = \theta + \xi_a$, then

$$(16)\quad \Delta\Pi_a = \Theta - \delta_a \,.$$

Introduce the notation:

$$\begin{aligned} \widehat{dz_i} &= (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (\widehat{dz_i} \wedge d\bar{z}_i) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n) \,, \\ \widehat{d\bar{z}_i} &= (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_i \wedge \widehat{d\bar{z}_i}) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n) \,. \end{aligned}$$

Then

$$\begin{aligned} {*dz_i} &= -2\left(\frac{\sqrt{-1}}{2}\right)^n \widehat{d\bar{z}_i} \,, \\ {*d\bar{z}_i} &= 2\left(\frac{\sqrt{-1}}{2}\right)^n \widehat{dz_i} \,. \end{aligned}$$

Define $\mu_a = \delta \Pi_a = -*d*\Pi_a$; then

$$(17) \qquad \mu_a = \delta\theta + \frac{1}{S_{2n}}\left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{\|z-a\|^{2n}} \Sigma_i \overline{(z_i - a_i)} \widehat{d\bar{z}_i} - (z_i - a_i)\widehat{dz_i} \ .$$

Since $d\mu_a = d\delta\Pi_a = \Delta\Pi_a = \Theta - \delta_a$, we have

$$(18) \qquad d\mu_a = \Theta \quad \text{in } C^n - \{a\} \ .$$

Since $\delta = \Lambda d^c - d^c\Lambda$, we have $\mu_a = \delta\Pi_a = d^c(-\Lambda\Pi_a)$. So define $\lambda_a = -\Lambda\Pi_a$, giving

$$(19) \qquad \mu_a = d^c\lambda_a \ .$$

We note also the explicit expression of λ_a (see (1)):

$$(20) \qquad \lambda_a = \frac{1}{4(n-1)\pi^n}\left[\left(1 - \frac{1}{(1+\|z\|)^{n-1}}\right) + \frac{1}{\|z-a\|^{2n-2}}\right]\omega_0^{n-1} \ .$$

It is important to remark that λ_a is strictly positive.

Before considering the general case of a holomorphic $f: V \to C^n$, where V is an open complex manifold of dimension n, we have to deal with the case of a holomorphic $f: D \to C^n$, where D is compact with boundary.

Theorem. *Let D be a compact complex manifold with boundary, and $f: D \to C^n$ holomorphic. Let $a \in C^n$ and $f^{-1}(a)$ be finite and disjoint from ∂D. Then*

$$(21) \qquad \int_D f^*\Theta = n(D, a) + \int_{\partial D} d^c f^*\lambda_a \ .$$

Remark. This is an analogue of the theorem in Part I [12]. The definitions and notation of Parts I and II are taken for granted. This theorem can be stated for a C^∞ $f: D \to R^d$, where D is compact, riemannian and of dimension d, but there is no need to treat this more general case.

Proof. Let U be an ε-neighborhood with respect to coordinate functions $\zeta_i = z_i - a_i$, $i = 1, \cdots, n$. Let $f^{-1}(a) = \{b_1, \cdots, b_p\}$, and U_i be a small ball neighborhood of b_i in $D - \partial D$ such that $b_i \in U_i$, $U_i \cap U_j = \emptyset$ if $i \neq j$, and $f(U_i) \subseteq U$ for all i. Then Stokes' Theorem implies that

$$\begin{aligned} \int_D f^*\Theta &= \lim_{\varepsilon\to 0} \int_{D-\{U_1,\cdots,U_p\}} f^*\Theta \\ &= \lim_{\varepsilon\to 0} \int_{D-\{U_1,\cdots,U_p\}} f^*d\mu_a \\ &= \int_{\partial D} f^*\mu_a - \lim_{\varepsilon\to 0} \Sigma_i \int_{\partial U_i} f^*\mu_a \\ &= \int_{\partial D} d^c f^*\lambda_a - \lim_{\varepsilon\to 0} \Sigma_i \int_{\partial U_i} f^*\mu_a \ , \end{aligned}$$

where we have used (18) and (19). To prove (21), it suffices to prove $n(D, a) = \lim_{\varepsilon\to 0} \Sigma_i \int_{\partial U_i} - f^*\mu_a$. Recall that if σ_i and σ are generators of $H_{2n-1}(U_i, U_i - \{b_i\})$ and $H_{2n-1}(U, U - \{a\})$ coherent with the orientation of D and C^n, respectively, and if $f_*(\sigma_i) = n_i\sigma$ ($n_i \in Z$), then $n(D, a) = \Sigma_i n_i$. The proof will be concluded with the demonstration of $n_i = \lim_{\varepsilon\to 0} \int_{\partial U_i} - f^*\mu_a$. Now suppose ν is a closed $(2n-1)$-form in $U - \{a\}$ such that $\int_\sigma \nu = 1$; then $f_*(\sigma_i) = n_i\sigma$ implies that $\int_{\sigma_i} f^*\nu = n_i$. Hence it remains to show $\lim_{\varepsilon\to 0} \int_{\partial U} - \mu_a = \int_{\partial U} \nu$ for some such ν. Now, from (17), we have that

$$\begin{aligned}\mu_a &= \delta\theta + \frac{1}{S_{2n}}\left(\frac{\sqrt{-1}}{2}\right)^n \left(\frac{1}{\|\xi\|^{2n}}\right) \Sigma_i\, \xi_i\, \widehat{d\bar\xi_i} - \xi_i\, \widehat{d\xi_i} \\ &\overset{\text{def}}{=} \delta\theta + \eta\ .\end{aligned}$$

Since $\delta\theta$ is C^∞ in the whole C^n, $\lim_{\varepsilon\to 0} \int_{\partial U} - \mu_a = \lim_{\varepsilon\to 0} \int_{\partial U} - \eta$. From (15), we know that $d(-\eta) = -d\delta\xi_a = \Delta\xi_a = 0$ in $C^n - \{a\}$. Hence, to terminate the proof, it suffices to show $\int_{\partial U'} - \eta = 1$ for all ε'-spheres $\partial U'$ in the $\zeta_1, \cdots, \zeta_n$ coordinate system ($\varepsilon' < \varepsilon$). But when restricted to $\partial U'$,

$$-\eta = \frac{-1}{S_{2n}}\left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{(\varepsilon')^{2n}} \Sigma_i(\bar\zeta_i\, \widehat{d\bar\zeta_i} - \zeta_i \widehat{d\zeta_i})\ .$$

So, to conclude that $\int_{\partial U'} - \eta = 1$, one merely has to observe that the volume of the ε'-sphere is $S_{2n}(\varepsilon')^{2n-1}$ and that $\frac{-2}{\varepsilon'}\left(\frac{\sqrt{-1}}{2}\right)^n \Sigma_i\, \bar\zeta_i\, \widehat{d\bar\zeta_i}$ and $\frac{2}{\varepsilon'}\left(\frac{\sqrt{-1}}{2}\right)^n \Sigma_i\, \zeta_i\, \widehat{d\zeta_i}$ are both volume elements of the ε'-sphere. q.e.d.

Suppose now $f\colon V \to C^n$ is holomorphic, where V is an open complex manifold of dimension n. Let τ be the exhaustion function on V, with the notation $V[t] = \tau^{-1}[0, t]$, $\partial V[t] = \tau^{-1}[t]$. We may apply the preceding to each $V[t]$, and the reasoning in § 2 of Part II carries over verbatim. Therefore, for each regular value $a \in C^n$ of f,

$$(22)\qquad \int_{r_0}^{r} n(t, a)dt = \int_{r_0}^{r} dt \int_{V[t]} f^*\Theta + S(r, a) + \int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a\ ,$$

where $n(t, a) = n(V[t], a)$, and

$$S(r, a) = \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau - \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a \,.$$

Let R denote the set $f(V) \subseteq C^n$. If $C^n - R$ has positive measure, then $\int_{C^n - R} \Theta = \varepsilon > 0$ because Θ is everywhere strictly positive. So (14) implies

$$\int_R \Theta = 1 - \varepsilon \,.$$

We shall integrate (22) with respect to Θ over R. In the sequel, the subscript a in Θ_a will denote the variable of integration. Equation (8) of Part II [13] has an analogue:

$$\int_R \left(\int_{r_0}^r n(t, a)dt\right)\Theta_a = \int_{r_0}^r dt \int_{V[t]} f^*\Theta \,. \tag{23}$$

It is furthermore obvious that:

$$\int_R \left(\int_{r_0}^r dt \int_{V[t]} f^*\Theta\right)\Theta_a = (1 - \varepsilon)\int_{r_0}^r dt \int_{V[t]} f^*\Theta \,. \tag{24}$$

Now assume in addition that V has a *convex exhaustion* τ. As in §§ 4 and 5 of Part II [13], all the measures which will show up are positive, and all the functions are positive and measurable with respect to each measure. Fubini's theorem then justifies the formal manipulations in the following. Thus,

$$\int_R S(r, a)\Theta_a \leq \int_R \left(\int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau\right)\Theta_a \leq \int_{C^n} \left(\int_{V[r]} f^*\lambda_a \wedge dd^c\tau\right)\Theta_a \,.$$

Let

$$p(z) = \left(1 - \frac{1}{(1 + \| z \|)^{n-1}}\right) \geq 0 \,,$$

$$q_a(z) = \frac{1}{\| z - a \|^{2n-2}} \geq 0 \,.$$

Then

$$\lambda_a = \frac{1}{4(n-1)\pi^n}[p(z) + q_a(z)]\omega_0^{n-1}$$

by (20). Hence,

$$\int_R S(r, a)\Theta_a \leq \int_{V[r]} f^*\omega_0^{n-1} \wedge dd^c\tau\left(\int_{C^n} \frac{1}{4(n-1)\pi^n} f^*p(z)\Theta_a\right)$$
$$+ \int_{V[r]} f^*\omega_0^{n-1} \wedge dd^c\tau\left(\int_{C^n} \frac{1}{4(n-1)\pi^n} f^*q_a(z)\Theta_a\right)$$
$$\leq \frac{1}{4(n-1)\pi^n} \int_{V[r]} (f^*p)f^*\omega_0^{n-1} \wedge dd^c\tau$$
$$+ \frac{1}{4(n-1)\pi^n} \int_{V[r]} (f^*\omega_0^{n-1} \wedge dd^c\tau)f^*\left(\int_{C^n} q_a(z)\Theta_a\right) .$$

Lemma. $g(z) \equiv \int_{C^n} q_a(z)\Theta_a$ *is a bounded function.*

Proof. It is well-known that such a function is continuous, so it suffices to prove that it is bounded outside of the compact set $B_2 \equiv \{\|z\| \leq 2\}$. So fix a $z \in C^n - B_2$, and let β be the ball of unit radius around z, i.e., $\beta = (a: \|a - z\| \leq 1\}$. Then $\|a - z\| > 1$ for all $a \in C^n - \beta$, so that

$$g(z) = \int_{C^n-\beta} \frac{n!}{\pi^n} \frac{\Psi_0(a)}{\|a - z\|^{2n-2}(1 + \|a\|^2)^{n+1}}$$
$$+ \int_\beta \frac{n!}{\pi^n} \frac{\Psi_0(a)}{\|a - z\|^{2n-2}(1 + \|a\|^2)^{n+1}}$$
$$\leq \int_{C^n} \frac{n!}{\pi^n} \frac{\Psi_0(a)}{(1 + \|a\|^2)^{n+1}} + \int_\beta \frac{n!}{\pi^n} \frac{\Psi_0(a)}{\|a - z\|^{2n-2}(1 + \|a\|^2)^{n+1}} .$$

Now $a \in \beta$ implies $\|a\| \geq 1$; consequently $a \in \beta$ implies $(1 + \|a\|^2)^{n+1} \geq (1 + \|a - z\|^2)^{n+1}$. Thus,

$$\int_\beta \frac{n!}{\pi^n} \frac{\Psi_0(a)}{\|a - z\|^{2n-2}(1 + \|a\|^2)^{n+1}} \leq \int_\beta \frac{n!}{\pi^n} \frac{\Psi_0(a)}{\|a - z\|^{2n-2}(1 + \|a - z\|^2)^{n+1}}$$
$$= \int_{B_1} \frac{n!}{\pi^n} \frac{\Psi_0(a)}{\|a\|^{2n-2}(1 + \|a\|^2)^{n+1}} \leq \int_{B_1} \frac{n!}{\pi^n} \frac{\Psi_0(a)}{(1 + \|a\|^2)^{n+1}} ,$$

where B_1 is the unit ball of C^n. Therefore,

$$g(z) \leq 2 \int_{C^n} \frac{n!}{\pi^n} \frac{\Psi_0(a)}{(1 + \|a\|^2)^{n+1}} < \infty . \qquad \text{q.e.d.}$$

So the lemma implies that there is a constant C, independent of r and a, such that

$$\int_R S(r, a)\Theta_a \leq \frac{1}{4(n-1)\pi^n}\left(\int_{V[r]} (f^*p)f^*\omega_0^{n-1} \wedge dd^c\tau + C\int_{V[r]} f^*\omega_0^{n-1} \wedge dd^c\tau\right).$$

As $p(z) \leq 1$, we see that for a constant C' which is independent of a and r, the following holds:

$$\int_R S(r, a)\Theta_a \leq C' \int_{V[r]} f^*\omega_0^{n-1} \wedge dd^c\tau . \tag{25}$$

In a similar fashion,

$$\int_R \left(\int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a\right)\Theta_a \leq C'' , \tag{26}$$

where C'' is a constant independent of r and a. Combining (22), $\cdots$, (26), we have,

$$\int_{r_0}^{r} dt \int_{V[t]} f^*\Theta \leq (1-\varepsilon)\int_{r_0}^{r} dt \int_{V[t]} f^*\Theta + C'\int_{V[r]} f^*\omega_0^{n-1} \wedge dd^c\tau + C'' .$$

Hence, as an immediate consequence,

Theorem. *Let $f: V \to C^n$ be holomorphic, where V is an n-dimensional complex manifold admitting a convex exhaustion. If df is nonsingular somewhere, and for Θ as in* (13),

$$\liminf_{r\to\infty} \frac{\int_{V[r]} f^*\omega_0^{n-1} \wedge dd^c\tau}{\int_{r_0}^{r} dt \int_{V[t]} f^*\Theta} = 0 , \tag{27}$$

then f is quasi-surjective.

Remark. Suppose that $V = C^n$, and $dd^c\tau_0 = 4\omega_0$ as in (2). Then (27) will be satisfied if, roughly speaking, f grows exponentially in all of its component functions. In this sense, this theorem is of the same order of strength as Corollary 5.4 of Part II [13].

References

(*Continuation of Part* II [13])

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